

# Multi-Head Finite Automata: Characterizations, Concepts and Open Problems

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Multi-head finite automata were introduced in [36] and [38]. Since that time, a vast literature on computational and descriptonal complexity issues on multi-head finite automata documenting the importance of these devices has been developed. Although multi-head finite automata are a simple concept, their computational behavior can be already very complex and leads to undecidable or even non-semi-decidable problems on these devices such as, for example, emptiness, finiteness, universality, equivalence, etc. These strong negative results trigger the study of subclasses and alternative characterizations of multi-head finite automata for a better understanding of the nature of non-recursive trade-offs and, thus, the borderline between decidable and undecidable problems. In the present paper, we tour a fragment of this literature.

## 1 Introduction

Languages accepted by multi-tape or multi-head finite automata were introduced in [36] and [38]. Since that time, many restrictions and generalizations of the original models were investigated and studied (see, e.g., [47]). Two-way deterministic and nondeterministic *multi-head* finite automata are probably best known to characterize the complexity classes of deterministic and nondeterministic logarithmic space. In fact, in [6] it was shown that the question of the equality of deterministic and nondeterministic logarithmic space is equivalent to the question of whether every language accepted by some nondeterministic two-way three-head finite automaton is accepted by some deterministic two-way multi-head finite automaton. Later this result was improved in [44]. Deterministic and nondeterministic one- and two-way multi-head finite automata induce strict hierarchies of language families with respect to the number of heads [28, 48].

Although multi-head finite automata are very simple devices, their computational behavior is already highly complex. But what about the size of multi-head finite automata opposed to their computational power? Questions on the economics of description size were already investigated in the early days of theoretical computer science and build a cornerstone of descriptonal complexity theory [27, 35, 42]. In terms of descriptonal complexity, a known upper bound for the trade-off between different descriptonal systems answers the question, how succinctly a language can be represented by a descriptor of one descriptonal system compared to an equivalent description of another descriptonal system. When dealing with more complicated devices such as, for example, pushdown automata that accept regular languages, a qualitative phenomenon revealed, namely that of non-recursive trade-offs. There the gain in economy of description can be arbitrary, that is, there are no recursive functions serving as upper bounds for the trade-off. This phenomenon was first discovered in [27], and in the meantime, a lot of deep and inter-

esting results on non-recursive trade-offs have been found for powerful enough computational devices almost everywhere (see, e.g., [4, 7, 8, 21, 22, 40, 46]).

Concerning descriptonal complexity in connection with multi-head finite automata, in [22] it is shown that the simulation of  $1\text{NFA}(2)$  by  $1\text{DFA}(k)$ , for  $k \geq 2$ , causes non-recursive trade-offs. Similar results hold for two-way devices as well [18]. An immediate consequence from the proofs of these results is that almost all of the aforementioned elementary questions become undecidable—in fact they are even shown to be not semi-decidable. Furthermore, because of these non-recursive trade-offs pumping lemmas and minimization algorithms for the automata in question do not exist. These strong negative results trigger the study of subclasses of multi-head finite automata for a better understanding of the nature of non-recursive trade-offs and, thus, the borderline between decidable and undecidable problems. From the legion of possible research directions we focus on three alternative and intermediate computational models, namely (i) multi-head automata accepting bounded languages, (ii) data-independent or oblivious multi-head finite automata, and (iii) parallel communicating finite automata. While the former research on bounded languages dates back to [3], the latter two topics were recently investigated in [9, 10] and [1]. In fact, for some of these models, some of the aforementioned elementary questions turn out to become decidable, while for others undecidability remains. At this point it is worth mentioning, that recently it was shown that even stateless one-way multi-head finite automata have a non-decidable emptiness problem [14]. In fact, these devices are the most simple one, since they have one internal state only.

In the present paper we tour a fragment of the literature on computational and descriptonal complexity issues of multi-head finite automata. It obviously lacks completeness, as one falls short of exhausting the large selection of multi-head finite automata related problems considered in the literature. We give our view of what constitute the most recent interesting links to the considered problem areas.

## 2 Multi-Head Finite Automata

We denote the set of non-negative integers by  $\mathbb{N}$ . We write  $\Sigma^*$  for the set of all words over the finite alphabet  $\Sigma$ . The empty word is denoted by  $\lambda$ , and  $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$ . The reversal of a word  $w$  is denoted by  $w^R$  and for the length of  $w$  we write  $|w|$ . We use  $\subseteq$  for inclusions and  $\subset$  for strict inclusions. We write  $2^S$  for the powerset of a set  $S$ .

Let  $k \geq 1$  be a natural number. A two-way  $k$ -head finite automaton is a finite automaton having a single read-only input tape whose inscription is the input word in between two endmarkers. The  $k$  heads of the automaton can move freely on the tape but not beyond the endmarkers. A formal definition is:

**Definition 1.** A nondeterministic two-way  $k$ -head finite automaton ( $2\text{NFA}(k)$ ) is a system  $\langle S, A, k, \delta, \triangleright, \triangleleft, s_0, F \rangle$ , where  $S$  is the finite set of internal states,  $A$  is the set of input symbols,  $k \geq 1$  is the number of heads,  $\triangleright \notin A$  and  $\triangleleft \notin A$  are the left and right endmarkers,  $s_0 \in S$  is the initial state,  $F \subseteq S$  is the set of accepting states, an  $\delta$  is the partial transition function mapping  $S \times (A \cup \{\triangleright, \triangleleft\})^k$  into the subsets of  $S \times \{-1, 0, 1\}^k$ , where 1 means to move the head one square to the right,  $-1$  means to move it one square to the left, and 0 means to keep the head on the current square. Whenever  $(s', (d_1, \dots, d_k)) \in \delta(s, (a_1, \dots, a_k))$  is defined, then  $d_i \in \{0, 1\}$  if  $a_i = \triangleright$ , and  $d_i \in \{-1, 0\}$  if  $a_i = \triangleleft$ ,  $1 \leq i \leq k$ .

A  $2\text{NFA}(k)$  starts with all of its heads on the first square of the tape. It halts when the transition function is not defined for the current situation. A configuration of a  $2\text{NFA}(k)$   $M = \langle S, A, k, \delta, \triangleright, \triangleleft, s_0, F \rangle$  at some time  $t$  with  $t \geq 0$  is a triple  $c_t = (w, s, p)$ , where  $w$  is the input,  $s \in S$  is the current state, and  $p = (p_1, \dots, p_k) \in \{0, \dots, |w| + 1\}^k$  gives the current head positions. If a position  $p_i$  is 0, then

head  $i$  is scanning the symbol  $\triangleright$ , if it is  $n+1$ , then the head is scanning the symbol  $\triangleleft$ . The initial configuration for input  $w$  is set to  $(w, s_0, (1, \dots, 1))$ . During its course of computation,  $M$  runs through a sequence of configurations. One step from a configuration to its successor configuration is denoted by  $\vdash$ . Let  $w = a_1 a_2 \dots a_n$  be the input,  $a_0 = \triangleright$ , and  $a_{n+1} = \triangleleft$ , then we set  $(w, s, (p_1, \dots, p_k)) \vdash (w, s', (p_1 + d_1, \dots, p_k + d_k))$  if and only if  $(s', (d_1, \dots, d_k)) \in \delta(s, (a_{p_1}, \dots, a_{p_k}))$ . As usual we define the reflexive, transitive closure of  $\vdash$  by  $\vdash^*$ . Note, that due to the restriction of the transition function, the heads cannot move beyond the endmarkers.

The language accepted by a  $2NFA(k)$  is precisely the set of words  $w$  such that there is some computation beginning with  $\triangleright w \triangleleft$  on the input tape and ending with the  $2NFA(k)$  halting in an accepting state, i.e.,  $L(M) = \{w \in A^* \mid (w, s_0, (1, \dots, 1)) \vdash^* (w, s, (p_1, \dots, p_k)), s \in F, \text{ and } M \text{ halts in } (w, s, (p_1, \dots, p_k))\}$ . If in any case  $\delta$  is either undefined or a singleton, then the  $k$ -head finite automaton is said to be *deterministic*. Deterministic two-way  $k$ -head finite automata are denoted by  $2DFA(k)$ . If the heads never move to the left, then the  $k$ -head finite automaton is said to be *one-way*. Nondeterministic and deterministic one-way  $k$ -head finite automata are denoted by  $1NFA(k)$  and  $1DFA(k)$ . The family of all languages accepted by a device of some type  $X$  is denoted by  $\mathcal{L}(X)$ .

The power of multi-head finite automata is well studied in the literature. For one-head machines we obtain a characterization of the family of regular languages REG. A natural question is to what extent the computational power depends on the number of heads. For (one-way) automata the proper inclusion  $\mathcal{L}(1NFA(1)) \subset \mathcal{L}(1DFA(2))$  is evident. An early result is the inclusion which separates the next level, that is,  $\mathcal{L}(1DFA(2)) \subset \mathcal{L}(1DFA(3))$  [15]. The breakthrough occurred in [48], where it was shown that the language  $L_n = \{w_1 \$ w_2 \$ \dots \$ w_{2n} \mid w_i \in \{a, b\}^* \text{ and } w_i = w_{2n+1-i} \text{ for } 1 \leq i \leq n\}$  can be used to separate the computational power of automata with  $k+1$  heads from those with  $k$  heads in the one-way setting:

**Theorem 2.** *Let  $k \geq 1$ . Then  $\mathcal{L}(1DFA(k)) \subset \mathcal{L}(1DFA(k+1))$  and  $\mathcal{L}(1NFA(k)) \subset \mathcal{L}(1NFA(k+1))$ .*

By exploiting the same language, the computational power of nondeterministic classes could be separated from the power of deterministic classes. To this end, for any  $n$  the complement of  $L_n$  was shown to be accepted by some one-way two-head nondeterministic finite automaton. Since the deterministic language families  $\mathcal{L}(1DFA(k))$  are closed under complementation, it follows that the inclusions  $\mathcal{L}(1DFA(k)) \subset \mathcal{L}(1NFA(k))$  are proper, for all  $k \geq 2$ . In order to compare one- and two-way multi-head finite automata classes, let  $L = \{w \mid w \in \{a, b\}^* \text{ and } w = w^R\}$  be the mirror language. It is well known that the mirror language is not accepted by any  $1NFA(k)$ , but its complement belongs to  $\mathcal{L}(1NFA(2))$ . The next corollary summarizes the inclusions.

**Corollary 3.** *Let  $k \geq 2$ . Then  $\mathcal{L}(1DFA(k)) \subset \mathcal{L}(2DFA(k))$ ,  $\mathcal{L}(1DFA(k)) \subset \mathcal{L}(1NFA(k))$ , and  $\mathcal{L}(1NFA(k)) \subset \mathcal{L}(2NFA(k))$ .*

From the complexity point of view, the two-way case is the more interesting one, since there is the following strong relation to the computational complexity classes  $L = DSPACE(\log(n))$  and  $NL = NSPACE(\log(n))$  [6].

**Theorem 4.**  $L = \bigcup_{k \geq 1} \mathcal{L}(2DFA(k))$  and  $NL = \bigcup_{k \geq 1} \mathcal{L}(2NFA(k))$ .

Concerning a head hierarchy of *two-way* multi-head finite automata, in [28] it was shown that  $k+1$  heads are better than  $k$ . Moreover, the witness languages are *unary*.

**Theorem 5.** *Let  $k \geq 1$ . Then there are unary languages that show the inclusions  $\mathcal{L}(2DFA(k)) \subset \mathcal{L}(2DFA(k+1))$  and  $\mathcal{L}(2NFA(k)) \subset \mathcal{L}(2NFA(k+1))$ .*

Whether nondeterminism is better than determinism in the two-way setting, is an open problem. In fact, in [44] it was shown that the equality for at least one  $k \geq 2$  implies  $L = NL$ . More generally,

$L = NL$  if and only if  $\mathcal{L}(1NFA(2)) \subseteq \bigcup_{k \geq 1} \mathcal{L}(2DFA(k))$  was shown in [44]. Due to a wide range of relations between several types of finite automata with different resources, the results and open problems for  $k$ -head finite automata apply in a similar way for other types. Here, we mention deterministic two-way finite automata with  $k$  pebbles ( $2DPA(k)$ ), with  $k$  linearly bounded counters ( $2DBCA(k)$ ), and with  $k$  linearly bounded counters with full-test ( $2DBCFA(k)$ ). In order to adapt the results, we present the hierarchy  $\mathcal{L}(2DFA(k)) \subseteq \mathcal{L}(2DPA(k)) \subseteq \mathcal{L}(2DBCA(k)) \subseteq \mathcal{L}(2DBCFA(k)) \subseteq \mathcal{L}(2DFA(k+1))$  which has been shown in several famous papers, e.g., [29, 33, 37, 45].

### 3 Descriptive Complexity

It is natural to investigate the succinctness of the representations of formal languages by different models. For example, it is well known that two-way and one-way finite automata are equivalent. Recently, the problem of the costs in terms of states for these simulations was solved in [19] by establishing a tight bound of  $\binom{2n}{n+1}$  for the simulation of two-way deterministic as well as nondeterministic finite automata by one-way nondeterministic finite automata. In the same paper tight bounds of  $n(n^n - (n-1)^n)$  and  $\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \binom{n}{i} \binom{n}{j} (2^i - 1)^j$  are shown for two-way deterministic and two-way nondeterministic finite automata simulations by one-way deterministic finite automata. Nevertheless, some challenging problems of finite automata are still open. An important example is the question of how many states are sufficient and necessary to simulate  $2NFA(1)$  with  $2DFA(1)$ ; we refer to [12, 23, 39, 41] for further reading. All trade-offs mentioned so far are bounded by recursive functions. But, for example, there is no recursive function which bounds the savings in descriptive complexity between deterministic and unambiguous pushdown automata [46]. A survey of the phenomenon of non-recursive trade-offs is [22]. Here we ask for the descriptive complexity of  $k$ -head finite automata. How succinctly can a language be presented by a  $k$ -head finite automaton compared with the presentation by a nondeterministic pushdown automaton, or by a  $(k+1)$ -head finite automaton?

**Definition 6.** A descriptive system  $E$  is a recursive set of finite descriptors, where each descriptor  $D \in E$  describes a formal language  $L(D)$ , and there exists an effective procedure to convert  $D$  into a Turing machine that decides (semi-decides) the language  $L(D)$ , if  $L(D)$  is recursive (recursively enumerable). The family of languages represented (or described) by some descriptive system  $E$  is  $L(E) = \{L(D) \mid D \in E\}$ . For every language  $L$ , we define  $E(L) = \{D \in E \mid L(D) = L\}$ .

A complexity (size) measure for  $E$  is a total, recursive function  $c : E \rightarrow \mathbb{N}$ , such that for any alphabet  $A$ , the set of descriptors in  $E$  describing languages over  $A$  is recursively enumerable in order of increasing size, and does not contain infinitely many descriptors of the same size.

Let  $E_1$  and  $E_2$  be two descriptive systems, and  $c$  be a complexity measure for  $E_1$  and  $E_2$ . A function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , with  $f(n) \geq n$ , is said to be an upper bound for the increase in complexity when changing from a minimal description in  $E_1$  to an equivalent minimal description in  $E_2$ , if for all  $L \in L(E_1) \cap L(E_2)$  we have  $\min\{c(D) \mid D \in E_2(L)\} \leq f(\min\{c(D) \mid D \in E_1(L)\})$ . If there is no recursive upper bound, the trade-off is said to be non-recursive.

In connection with multi-head finite automata the question of determining the trade-offs between the levels of the head hierarchies arises immediately. Recently, the problem whether these trade-offs are non-recursive has been solved for two-way devices in the affirmative [18] (cf. also [21]):

**Theorem 7.** Let  $k \geq 1$ . Then the trade-off between deterministic (nondeterministic) two-way  $(k+1)$ -head finite automata and deterministic (nondeterministic) two-way  $k$ -head finite automata is non-recursive. Moreover, the results hold also for automata accepting unary languages.

But how to prove non-recursive trade-offs? Roughly speaking, most of the proofs appearing in the literature are basically relying on one of two different schemes. One of these techniques is due to Hartmanis [7, 8]. Next, we present a slightly generalized and unified form of this technique [21, 22].

**Theorem 8.** *Let  $E_1$  and  $E_2$  be two descriptive systems for recursive languages. If there exists a descriptive system  $E_3$  such that, given an arbitrary  $M \in E_3$ , (i) there exists an effective procedure to construct a descriptor in  $E_1$  for some language  $L_M$ , (ii)  $L_M$  has a descriptor in  $E_2$  if and only if  $L(M)$  does not have a property  $P$ , and (iii) property  $P$  is not semi-decidable for languages with descriptors in  $E_3$ , then the trade-off between  $E_1$  and  $E_2$  is non-recursive.*

In order to apply Theorem 8 one needs a descriptive system  $E_3$  with appropriate problems that are not even semi-decidable. An immediate descriptive system is the set of Turing machines for which only trivial problems are decidable and a lot of problems are not semi-decidable. Basically, we consider *valid computations of Turing machines* [5]. Roughly speaking, these are histories of accepting Turing machine computations. It suffices to consider deterministic Turing machines with one single tape and one single read-write head. Without loss of generality and for technical reasons, we assume that the Turing machines can halt only after an odd number of moves, accept by halting, make at least three moves, and cannot print a blank. A valid computation is a string built from a sequence of configurations passed through during an accepting computation. Let  $S$  be the state set of some Turing machine  $M$ , where  $s_0$  is the initial state,  $T \cap S = \emptyset$  is the tape alphabet containing the blank symbol,  $A \subset T$  is the input alphabet, and  $F \subseteq S$  is the set of accepting states. Then a configuration of  $M$  can be written as a word of the form  $T^*ST^*$  such that  $t_1 \dots t_i s t_{i+1} \dots t_n$  is used to express that  $M$  is in state  $s$ , scanning tape symbol  $t_{i+1}$ , and  $t_1$  to  $t_n$  is the support of the tape inscription.

For the purpose of the following, valid computations  $\text{VALC}(M)$  are now defined to be the set of words  $\$w_1\$w_2\$ \dots \$w_{2n}\$,$  where  $\$ \notin T \cup S$ ,  $w_i \in T^*ST^*$  are configurations of  $M$ ,  $w_1$  is an initial configuration of the form  $s_0A^*$ ,  $w_{2n}$  is an accepting configuration of the form  $T^*FT^*$ , and  $w_{i+1}$  is the successor configuration of  $w_i$ , for  $1 \leq i < 2n$ . The set of *invalid computations*  $\text{INVALC}(M)$  is the complement of  $\text{VALC}(M)$  with respect to the alphabet  $\{\$ \} \cup T \cup S$ .

Now we complement the results on the trade-offs between the levels of the head hierarchy for two-way automata. We show non-recursive trade-offs between the levels of the head hierarchies of deterministic as well as nondeterministic one-way devices. Moreover, non-recursive trade-offs are shown between nondeterministic two-head and deterministic  $k$ -head automata.

**Example 9.** *Let  $M$  be a Turing machine. Then an  $\text{IDFA}(2)$   $M'$  can be constructed that accepts  $\text{VALC}(M)$ . One task of  $M'$  is to verify the correct form of the input, that is, whether it is of the form  $\$s_0A^*\$T^*ST^*\$ \dots \$T^*ST^*\$T^*FT^*\$$ . This task means to verify whether the input belongs to a regular set and can be done in parallel to the second task.*

*The second task is to verify for each two adjacent subwords whether the second one represents the successor configuration of the first one. We show the construction for  $w_i\$w_{i+1}$ . Starting with the first head on the first symbol of  $w_i$  and the second head on the first symbol of  $w_{i+1}$ , automaton  $M'$  compares the subwords symbolwise by moving the heads to the right. Turing machine  $M$  has three possibilities to move its head. So,  $w_i = t_1 \dots t_i s t_{i+1} \dots t_n$  goes to  $t_1 \dots t_i s' t'_{i+1} \dots t_n$ , to  $t_1 \dots s' t'_{i+1} \dots t_n$ , or to  $t_1 \dots t'_{i+1} s' \dots t_n$ . Each of the three possibilities can be detected by  $M'$ . Furthermore,  $M'$  can verify whether the differences between  $w_i$  and  $w_{i+1}$  are due to a possible application of the transition function of  $M$ . Finally, the first head is moved on the first symbol of  $w_{i+1}$ , and the second head is moved on the first symbol of  $w_{i+2}$  to start the verification of  $w_{i+1}$  and  $w_{i+2}$ .*

**Theorem 10.** *Let  $k \geq 2$ . The trade-off between  $\text{IDFA}(k)$  and nondeterministic pushdown automata is non-recursive.*



*Proof.* In order to apply Theorem 8, let  $E_3$  be the set of Turing machines. For every  $M \in E_3$ , define  $L_M$  to be  $\text{VALC}(M)$ . So,  $L_M$  is accepted by some  $\text{IDFA}(k)$ . In [5] it was shown that  $\text{VALC}(M)$  is context free if and only if  $L(M)$  is finite. Since infiniteness is not semi-decidable for Turing machines, all conditions of Theorem 8 are satisfied and the assertion follows.  $\square$

**Corollary 11.** *Let  $k \geq 2$ . The trade-off between  $\text{IDFA}(k)$  and  $\text{INFA}(1)$  is non-recursive.*

In the following we exploit the basic hierarchy theorem shown in [48], and recall that the language  $L_n = \{w_1\$w_2\$ \dots \$w_{2n} \mid w_i \in \{a, b\}^* \text{ and } w_i = w_{2n+1-i} \text{ for } 1 \leq i \leq n\}$  is accepted by some deterministic or nondeterministic one-way  $k$ -head finite automaton if and only if  $n \leq \binom{k}{2}$ . Now we extend the language in order to meet our purposes. Basically, the idea is to keep the structure of the language but to build the subwords  $w_i$  over an alphabet of pairs, that is,  $w_i = \begin{smallmatrix} u_1 & u_2 & \dots & u_m \\ v_1 & v_2 & \dots & v_m \end{smallmatrix}$ , where the  $u_j$  and  $v_j$  are symbols such that the upper parts of the subwords are words over  $\{a, b\}$  as in  $L_n$ . The lower parts are valid computations of some given Turing machine  $M$ . Let  $W_M = \{ \begin{smallmatrix} u \\ v \end{smallmatrix} \mid u \in \{a, b\}^*, v \in \text{VALC}(M), |u| = |v| \}$ . Then  $L_{n,M}$  is defined to be  $\{w_1\$w_2\$ \dots \$w_{2n} \mid w_i \in W_M \text{ and } w_i = w_{2n+1-i} \text{ for } 1 \leq i \leq n\}$ .

**Lemma 12.** *Let  $k \geq 2$  and  $M$  be some Turing machine. Then a  $\text{IDFA}(k+1)$  can be constructed that accepts  $L_{\binom{k}{2}+1,M}$ .*

*Proof.* Let  $n = \binom{k}{2} + 1$ . We sketch the construction. At first, two of the heads, say the first and  $(k+1)$ st one, are used to verify that the lower parts of the subwords  $w_1, w_2, \dots, w_n$  do belong to  $\text{VALC}(M)$  (cf. Example 9). At the end of this task, both heads are positioned at the beginning of  $w_{n+1}$ . Next, the first head is moved to the beginning of  $w_{2n+1-(k-1)}$  while the remaining  $k-1$  heads are moved to the beginnings of  $w_1, w_2, \dots, w_{k-1}$ , respectively. Now the  $k-1$  words  $w_{2n+1-(k-1)}$  to  $w_{2n}$  are successively compared with the words  $w_{k-1}$  to  $w_1$ . Next, the  $k-1$  heads are moved to the beginning of  $w_k$ , and the same procedure is applied inductively to verify  $w_k\$w_{k+1}\$ \dots \$w_{2n+1-k}$ .

After  $k-1$  repetitions  $\sum_{i=k-1}^1 i = \frac{k^2-k}{2} = \binom{k}{2}$  pairs have been compared, that is, the pairs  $w_i$  and  $w_{2n+1-i}$ , for  $1 \leq i \leq n-1$ . So, one of the heads that compared  $w_{n-1}$  and  $w_{n+2}$  is positioned at the beginning of  $w_n$ . Since head  $k+1$  is still positioned at the beginning of  $w_{n+1}$ , the remaining pair  $w_n$  and  $w_{n+1}$  can be compared.  $\square$

It is a straightforward adaption of the hierarchy result of [48] to prove that the language  $L_{\binom{k}{2}+1,M}$  is not accepted by any  $\text{INFA}(k)$  if  $L(M)$  is infinite.

Now we can apply Theorem 8 in order to show the non-recursive trade-offs between any two levels of the deterministic or nondeterministic head hierarchy.

**Theorem 13.** *Let  $k \geq 1$ . The trade-offs between  $\text{IDFA}(k+1)$  and  $\text{IDFA}(k)$ , between  $\text{INFA}(k+1)$  and  $\text{INFA}(k)$ , and between  $\text{IDFA}(k+1)$  and  $\text{INFA}(k)$  are non-recursive.*

*Proof.* For  $k = 1$ , the theorem has been shown by Corollary 11. So, let  $k \geq 2$ . We apply Theorem 8 by setting  $E_3$  to be the set of Turing machines and  $L_M = L_{\binom{k}{2}+1,M}$ . As property  $P$  we choose infiniteness. Lemma 12 shows that the language  $L_M$  is accepted by some  $\text{IDFA}(k+1)$  and, therefore, by some  $\text{INFA}(k+1)$ . On the other hand, language  $L_M$  is not accepted by any  $\text{INFA}(k)$  and, therefore, not accepted by any  $\text{IDFA}(k)$  if  $L(M)$  is infinite. Conversely, if  $L(M)$  is finite, the set  $\text{VALC}(M)$  is finite. This implies that  $L_{\binom{k}{2}+1,M} = L_M$  is finite and, thus, is accepted by some  $\text{IDFA}(k)$  and, therefore, by some  $\text{INFA}(k)$ .  $\square$

The next question asks for the trade-offs between nondeterministic and deterministic automata. Clearly, the trade-off between 1NFA(1) and 1DFA(1) is recursive. But following an idea in [48] which separates nondeterminism from determinism, we can show non-recursive trade-offs on every level  $k \geq 2$  of the hierarchy as follows.

**Theorem 14.** *Let  $k \geq 2$ . Then the trade-off between 1NFA(2) and 1DFA( $k$ ) and, thus, between 1NFA( $k$ ) and 1DFA( $k$ ) is non-recursive.*

Another consequence of the fact that the set of valid computations is accepted by  $k$ -head finite automata is that many of their properties are not even semi-decidable. We can transfer the results from Turing machines.

**Theorem 15.** *Let  $k \geq 2$ . Then the problems of emptiness, finiteness, infiniteness, universality, inclusion, equivalence, regularity, and context-freeness are not semi-decidable for  $\mathcal{L}(1DFA(k))$ ,  $\mathcal{L}(1NFA(k))$ ,  $\mathcal{L}(2DFA(k))$ , and  $\mathcal{L}(2NFA(k))$ .*

**Theorem 16.** *Let  $k \geq 2$ . Then any language family whose word problem is semi-decidable and that effectively contains the language families  $\mathcal{L}(1DFA(k))$ ,  $\mathcal{L}(1NFA(k))$ ,  $\mathcal{L}(2DFA(k))$ , or  $\mathcal{L}(2NFA(k))$  does not possess a pumping lemma.*

Finally, note that no minimization algorithm for the aforementioned devices exists, since otherwise emptiness becomes decidable, which contradicts Theorem 15.

**Theorem 17.** *There is no minimization algorithm converting some 1DFA( $k$ ), 1NFA( $k$ ), 2DFA( $k$ ), or 2NFA( $k$ ), for  $k \geq 2$ , to an equivalent automaton of the same type with a minimal number of states.*

## 4 Alternative and Intermediate Computational Models

The existence of non-recursive trade-offs and the undecidability of many decidability problems for multi-head finite automata is at least disconcerting from an applied perspective. So, the question arises under which assumptions undecidable questions become decidable. Here we focus on three different alternative and intermediate computational models, namely (i) multi-head automata accepting bounded languages, (ii) data-independent or oblivious multi-head finite automata, and (iii) parallel communicating finite automata.

### 4.1 Multi-Head Finite Automata Accepting Bounded Languages

If we impose the structural restriction of “boundedness,” then for context-free grammars and pushdown automata it is known that the trade-offs become recursive and decidability questions become decidable [24]. In this section, we summarize corresponding results for multi-head finite automata accepting bounded languages.

**Definition 18.** *Let  $A = \{a_1, a_2, \dots, a_n\}$ . A language  $L \subseteq A^*$  is said to be letter-bounded or bounded if  $L \subseteq a_1^* a_2^* \dots a_n^*$ . A subset  $P \subseteq \mathbb{N}^n$  is said to be a linear set if there exist  $\alpha_0, \alpha_1, \dots, \alpha_m$  in  $\mathbb{N}^n$  such that  $P = \{\beta \mid \beta = \alpha_0 + \sum_{i=1}^m \alpha_i \alpha_i, \text{ where } \alpha_i \geq 0, \text{ for } 1 \leq i \leq m\}$  and  $P$  is said to be semilinear if it is the finite union of linear sets.*

In order to relate bounded languages and semilinear sets we introduce the following notation. For an alphabet  $A = \{a_1, a_2, \dots, a_n\}$  the Parikh mapping  $\Psi : \Sigma^* \rightarrow \mathbb{N}^n$  is defined by  $\Psi(w) = (|w|_{a_1}, |w|_{a_2}, \dots, |w|_{a_n})$ , where  $|w|_{a_i}$  denotes the number of occurrences of  $a_i$  in the word  $w$ . We say

that a language is *semilinear* if its Parikh image is a semilinear set. In [31] a fundamental result concerning the distribution of symbols in the words of a context-free language has been shown. It says that for any context-free language  $L$ , the Parikh image  $\Psi(L) = \{\Psi(w) \mid w \in L\}$  is semilinear. Semilinear sets have many appealing properties. For example, they are closed under union, intersection, and complementation [3]. Furthermore, bounded semilinear languages have nice decidability properties, since the questions of emptiness, universality, finiteness, infiniteness, inclusion, and equivalence are decidable (cf. [3]). The main connection between bounded languages and multi-head finite automata is the following result [13, 43].

**Lemma 19.** *Let  $k \geq 1$  and  $L$  be a bounded language accepted by a  $2NFA(k)$  which performs a constant number of head reversals. Then  $\Psi(L)$  is a semilinear set.*

The above characterization implies immediately that there are no one-way multi-head finite automata which accept the non-semilinear language  $L_1 = \{a^{2^n} \mid n \geq 1\}$ . The situation changes for unbounded languages. For example, the non-semilinear language  $L_2 = \{aba^2ba^3b \dots a^n b \mid n \geq 1\}$  is accepted by some  $1DFA(2)$  by checking that the size of adjacent  $a$ -blocks is always increasing by one. Due to the relation to semilinear sets and their positive decidability results we obtain the borderline between decidability and undecidability as follows.

**Theorem 20.** *Let  $k \geq 2$ . The problems of emptiness, universality, finiteness, infiniteness, inclusion, and equivalence are undecidable for the language families  $\mathcal{L}(1DFA(k))$ ,  $\mathcal{L}(1NFA(k))$ ,  $\mathcal{L}(2DFA(k))$ , and  $\mathcal{L}(2NFA(k))$ , if the automata are allowed to perform at most a constant number of head reversals. The problems become decidable for bounded languages.*

We next summarize results concerning decidability questions on models which are generalized or restricted versions of multi-head finite automata. Let us start with nondeterministic two-way multi-head pushdown automata [13] which are nondeterministic two-way multi-head finite automata augmented by a pushdown store. In fact, the characterization in Lemma 19 has been shown in [13] for the stronger model of nondeterministic two-way multi-head pushdown automata. So, for  $k \geq 1$  let a bounded language  $L$  be accepted by some nondeterministic two-way  $k$ -head pushdown automaton which performs a constant number of head reversals. Then  $\Psi(L)$  is a semilinear set.

Thus, the positive decidability results obviously hold also for such automata. As is the case for multi-head finite automata, we lose positive decidability and semilinearity if we drop the condition of boundedness. We may consider again the language  $L_2$  which is clearly accepted by a one-way two-head pushdown automaton as well.

In the general model of multi-head finite automata the moves of the heads are only depending on the input, and a head cannot feel the presence of another head at the same position at the same time. If this property is added to each head, we come to the model of one-way multi-head finite automata having *sensing heads* (see [13]). This model is stronger since, for example, non-semilinear language  $\{a^{n^2} \mid n \geq 1\}$  can be accepted by a deterministic one-way finite automaton having three sensing heads, whereas with non-sensing heads only semilinear languages are accepted in the unary case (see Lemma 19). On the other hand, it is shown in [13] that every bounded language which is accepted by some nondeterministic one-way finite automaton having two sensing heads can be also accepted by some nondeterministic one-way pushdown automaton with two heads, and thus is semilinear.

**Theorem 21.** *The problems of emptiness, universality, finiteness, infiniteness, inclusion, and equivalence are undecidable for deterministic one-way finite automata with at least three sensing heads. The aforementioned problems become decidable for nondeterministic one-way finite automata with two sensing heads that accept a bounded language.*



The computational capacity of two-way finite automata with sensing heads and bounds on the number of input reversals is investigated in [11]. We next consider a restricted version of one-way multi-head finite automata which allows only one designated head to read and distinguish input symbols whereas the remaining heads cannot distinguish input symbols, that is, the remaining heads get only the information whether they are reading an input symbol or the right endmarker. These automata are called *partially blind* multi-head finite automata [16]. In this model the decidability results are much better, since it can be shown that every language, not necessarily bounded, which is accepted by a partially blind nondeterministic one-way multi-head finite automaton has a semilinear Parikh map. In detail, we have the following theorem.

**Theorem 22** ([16]). *(1) The problems of emptiness, finiteness, and infiniteness are decidable for partially blind nondeterministic one-way multi-head finite automata. Universality is undecidable and, thus, the problems of inclusion and equivalence are undecidable as well. (2) The language family accepted by partially blind deterministic one-way multi-head finite automata is closed under complementation. Thus, the problems of universality, inclusion, and equivalence are decidable. (3) The problems of emptiness, universality, finiteness, infiniteness, inclusion, and equivalence are decidable for partially blind nondeterministic one-way multi-head finite automata that accept a bounded language.*

## 4.2 Data-Independent Multi-Head Finite Automata

The notion of oblivious Turing programs was introduced in [32]. Recently, obliviousness was investigated in relation with computational models from classical automata theory as, for example, multi-head finite automata in [10].

**Definition 23.** *A  $k$ -head finite automaton  $M$  is data-independent or oblivious if the position of every input-head  $i$  after step  $t$  in the computation on input  $w$  is a function  $f_M(|w|, i, t)$  that only depends on  $i$ ,  $t$ , and  $|w|$ .*

We denote deterministic (nondeterministic) one-way and two-way data-independent  $k$ -head finite automata by  $1\text{DiDFA}(k)$  and  $2\text{DiDFA}(k)$  ( $1\text{DiNFA}(k)$  and  $2\text{DiNFA}(k)$ ). In [34] it was shown that data-independence is no restriction for multi-head one-counter automata, multi-head non-erasing stack automata, and multi-head stack automata. The same is obviously true for one-head finite automata.

Interestingly, one can show that for data-independent  $k$ -head finite automata determinism is as powerful as nondeterminism. To this end, observe that a data-independent automaton with  $k$  heads has, on inputs of length  $n$ , only one possible input-head trajectory during a computation. So, the only nondeterminism left in the  $k$ -head nondeterministic finite automata is the way in which the next state is chosen. Therefore, a power-set construction shows that even data-independent determinism may simulate data-independent nondeterminism. Thus, we have the following theorem.

**Theorem 24.** *Let  $k \geq 1$ . Then  $\mathcal{L}(1\text{DiDFA}(k)) = \mathcal{L}(1\text{DiNFA}(k))$  and moreover  $\mathcal{L}(2\text{DiDFA}(k)) = \mathcal{L}(2\text{DiNFA}(k))$ .*

But why are data-independent multi-head finite automata that interesting? Similarly as ordinary multi-head finite automata characterize logarithmic space bounded Turing machine computations [6], data-independent multi-head finite automata capture the parallel complexity class  $\text{NC}^1$ , which is the set of problems accepted by log-space uniform circuit families of logarithmic depth, polynomial size, with AND- and OR-gates of bounded fan-in. We have  $\text{NC}^1 \subseteq \text{L}$ . For the characterization the following observation is useful: Every deterministic multi-head finite automaton that works on unary inputs is already data-independent by definition. Therefore,  $\text{L}^u \subseteq \bigcup_{k \geq 1} \mathcal{L}(2\text{DiDFA}(k))$ , where  $\text{L}^u$  denotes the set

of all unary languages from  $L$ . However, because of the trajectory argument, we do not know whether the inclusion  $NL^u \subseteq \bigcup_{k \geq 1} 2DiNFA(k)$  holds. A positive answer would imply  $NL^u \subseteq L^u$  by Theorem 24. This, in turn, would lead to a positive answer of the *linear bounded automaton (LBA) problem* by translational methods [20]. In [10] the following theorem was shown.

**Theorem 25.**  $NC^1 = \bigcup_{k \geq 1} \mathcal{L}(2DiDFA(k))$ .

Since there exists a log-space complete language in  $\mathcal{L}(1DFA(2))$ , we immediately obtain  $NC^1 = L$  if and only if  $\mathcal{L}(1DFA(2)) \subseteq \bigcup_{k \geq 1} \mathcal{L}(2DiDFA(k))$ . Similarly,  $NC^1 = NL$  if and only if  $\mathcal{L}(1NFA(2)) \subseteq \bigcup_{k \geq 1} \mathcal{L}(2DiDFA(k))$ .

What is known about the head hierarchies induced by data-independent multi-head finite automata? For two-way data-independent multi-head finite automata one can profit from known results [28]. Since for every  $k \geq 1$  there are unary languages that may serve as witnesses for the inclusions  $\mathcal{L}(2DFA(k)) \subset \mathcal{L}(2DFA(k+1))$  and  $\mathcal{L}(2NFA(k)) \subset \mathcal{L}(2NFA(k+1))$ , we obtain the following corollary.

**Corollary 26.** Let  $k \geq 1$ . Then  $\mathcal{L}(2DiDFA(k)) \subset \mathcal{L}(2DiDFA(k+1))$ .

The remaining inclusions  $\mathcal{L}(2DiDFA(k)) \subseteq \mathcal{L}(2DFA(k))$ , for  $k \geq 2$ , for two-way multi-head finite automata are related to the question of whether  $NC^1 = L$ . This parallels the issue of whether the inclusion  $\mathcal{L}(2DFA(k)) \subseteq \mathcal{L}(2NFA(k))$  is proper for  $k \geq 2$ . For the relationship between data-independent and data-dependent multi-head finite automata, we find that  $\mathcal{L}(2DiDFA(k)) = \mathcal{L}(2DFA(k))$ , for some  $k \geq 2$ , implies  $NC^1 = L$ . Moreover, it was shown in [9] that the head hierarchy for one-way data-independent multi-head finite automata is strict. Obviously,  $REG = \mathcal{L}(1DiDFA(1))$  and, hence,  $REG \subset \mathcal{L}(2DiDFA(2))$ . By adapting the head hierarchy result of [48] to data-independent automata the next theorem is shown.

**Theorem 27.** Let  $k \geq 1$ . Then  $\mathcal{L}(1DiDFA(k)) \subset \mathcal{L}(1DiDFA(\frac{k \cdot (k+1)}{2} + 3))$ .

Thus, we have an infinite proper hierarchy with respect to the number of heads, but the bound obtained in Theorem 27 is not very good, especially for small values of  $k$ . In fact, a separation for the first four levels was obtained in [9], using the language  $L_{F,n} = \{a^{i \cdot F(2)}\$a^{i \cdot F(3)}\$ \dots \$a^{i \cdot F(k+1)} \mid i \geq 1\}$ , where  $F(j)$  is the  $j$ th Fibonacci number. It holds  $L_{F,n} \in \mathcal{L}(1DiDFA(k))$ , if and only if  $n \leq \frac{k \cdot (k-1)}{2} + 1$ .

**Theorem 28.**  $\mathcal{L}(1DiDFA(1)) \subset \mathcal{L}(2DiDFA(2)) \subset \mathcal{L}(2DiDFA(3)) \subset \mathcal{L}(2DiDFA(4))$ .

Whether the one-way hierarchy for data-independent multi-head finite automata is strict in the sense that  $k+1$  heads are better than  $k$ , is an open problem. Concerning the remaining open inclusions the following is known.

**Theorem 29.** Let  $k \geq 2$ . Then  $\mathcal{L}(1DiDFA(k)) \subset \mathcal{L}(2DiDFA(k))$ .

Moreover, by the copy language as witness we find that one-way data-independent languages are a proper subset of two-way data-independent multi-head finite automata languages.

**Corollary 30.**  $\bigcup_{k \geq 1} \mathcal{L}(1DiDFA(k)) \subset \bigcup_{k \geq 1} \mathcal{L}(2DiDFA(k)) = NC^1$ .

We close this subsection by mentioning some open problems for further research: (1) Determine the bounds of the conversions of one-head data-independent finite automata into one-head data-dependent deterministic, nondeterministic, and alternating finite automata and *vice versa*. (2) Consider decidability and complexity questions such as equivalence, non-emptiness, etc. for  $k$ -head data-independent finite automata. Finally, the most interesting point for research might be (3) the one-way  $k$ -head hierarchy for data-independent finite automata. Is it a strict hierarchy, in the sense that  $k+1$  heads are better than  $k$ ?

### 4.3 Parallel Communicating Finite Automata

In this section we will focus on parallel communicating finite automata systems which were introduced in [25]. Basically, a parallel communicating finite automata system of degree  $k$  is a device of  $k$  finite automata working in parallel with each other on a common one-way read-only input tape and being synchronized according to a universal clock. The  $k$  automata communicate on request by states, that is, when some automaton enters a distinguished query state  $q_i$ , it is set to the current state of automaton  $A_i$ . Concerning the next state of the sender  $A_i$ , we distinguish two modes. In *non-returning* mode the sender remains in its current state whereas in *returning* mode the sender is set to its initial state. Moreover, we distinguish whether all automata are allowed to request communications, or whether there is just one master allowed to request communications. The latter types are called *centralized*.

**Definition 31.** A nondeterministic parallel communicating finite automata system of degree  $k$  ( $PCFA(k)$ ) is a construct  $\mathcal{A} = \langle \Sigma, A_1, A_2, \dots, A_k, Q, \triangleleft \rangle$ , where  $\Sigma$  is the set of input symbols, each  $A_i = \langle S_i, \Sigma, \delta_i, s_{0,i}, F_i \rangle$ , for  $1 \leq i \leq k$ , is a nondeterministic finite automaton with state set  $S_i$ , initial state  $s_{0,i} \in S_i$ , set of accepting states  $F_i \subseteq S_i$ , and transition function  $\delta_i : S_i \times (\Sigma \cup \{\lambda, \triangleleft\}) \rightarrow 2^{S_i}$ ,  $Q = \{q_1, q_2, \dots, q_k\} \subseteq \bigcup_{1 \leq i \leq k} S_i$  is the set of query states, and  $\triangleleft \notin \Sigma$  is the end-of-input symbol.

The automata  $A_1, A_2, \dots, A_k$  are called *components* of the system  $\mathcal{A}$ . A *configuration*  $(s_1, x_1, s_2, x_2, \dots, s_k, x_k)$  of  $\mathcal{A}$  represents the current states  $s_i$  as well as the still unread parts  $x_i$  of the tape inscription of all components  $1 \leq i \leq k$ . System  $\mathcal{A}$  starts with all of its components scanning the first square of the tape in their initial states. For input word  $w \in \Sigma^*$ , the initial configuration is  $(s_{0,1}, w\triangleleft, s_{0,2}, w\triangleleft, \dots, s_{0,k}, w\triangleleft)$ . Each step can consist of two phases. In a first phase, all components are in non-query states and perform an ordinary (non-communicating) step independently. The second phase is the communication phase during which components in query states receive the requested states as long as the sender is not in a query state itself. This process is repeated until all requests are resolved, if possible. If the requests are cyclic, no successor configuration exists. As mentioned above, we distinguish *non-returning* communication, that is, the sender remains in its current state, and *returning* communication, that is, the sender is reset to its initial state. For the first phase, we define the successor configuration relation  $\vdash$  by  $(s_1, a_1y_1, s_2, a_2y_2, \dots, s_k, a_ky_k) \vdash (p_1, z_1, p_2, z_2, \dots, p_k, z_k)$ , if  $Q \cap \{s_1, s_2, \dots, s_k\} = \emptyset$ ,  $a_i \in \Sigma \cup \{\lambda, \triangleleft\}$ ,  $p_i \in \delta_i(s_i, a_i)$ , and  $z_i = \triangleleft$  for  $a_i = \triangleleft$  and  $z_i = y_i$  otherwise, for  $1 \leq i \leq k$ . For non-returning communication in the second phase, we set  $(s_1, x_1, s_2, x_2, \dots, s_k, x_k) \vdash (p_1, x_1, p_2, x_2, \dots, p_k, x_k)$ , if, for all  $1 \leq i \leq k$  such that  $s_i = q_j$  and  $s_j \notin Q$ , we have  $p_i = s_j$ , and  $p_r = s_r$  for all the other  $r$ , for  $1 \leq r \leq k$ . Alternatively, for returning communication in the second phase, we set  $(s_1, x_1, s_2, x_2, \dots, s_k, x_k) \vdash (p_1, x_1, p_2, x_2, \dots, p_k, x_k)$ , if, for all  $1 \leq i \leq k$  such that  $s_i = q_j$  and  $s_j \notin Q$ , we have  $p_i = s_j$ ,  $p_j = s_{0,j}$ , and  $p_r = s_r$  for all the other  $r$ ,  $1 \leq r \leq k$ .

A computation *halts* when the successor configuration is not defined for the current situation. In particular, this may happen when cyclic communication requests appear, or when the transition function of one component is not defined. The language  $L(\mathcal{A})$  accepted by a  $PCFA(k)$   $\mathcal{A}$  is precisely the set of words  $w$  such that there is some computation beginning with  $w\triangleleft$  on the input tape and halting with at least one component having an undefined transition function and being in an accepting state. Let  $\vdash^*$  denote the reflexive and transitive closure of  $\vdash$  and set  $L(\mathcal{A}) = \{w \in \Sigma^* \mid (s_{0,1}, w\triangleleft, s_{0,2}, w\triangleleft, \dots, s_{0,k}, w\triangleleft) \vdash^* (p_1, a_1y_1, p_2, a_2y_2, \dots, p_k, a_ky_k), \text{ such that } p_i \in F_i \text{ and } \delta_i(p_i, a_i) \text{ is undefined, for some } 1 \leq i \leq k\}$ .

If all components  $A_i$  are deterministic finite automata, then the whole system is called *deterministic*, and we add the prefix D to denote it. The absence or presence of an R in the type of the system denotes whether it works in *non-returning* or *returning* mode, respectively. Finally, if there is just one component, say  $A_1$ , that is allowed to query for states, that is,  $S_i \cap Q = \emptyset$ , for  $2 \leq i \leq k$ , then the system is said to be

centralized. We denote centralized systems by a C. Whenever the degree is missing we mean systems of arbitrary degree. In order to clarify our notation we give an example.

**Example 32.** We consider the language  $\{w\$w \mid w \in \{a,b\}^+\}$  and show that it can be accepted by a DCPCFA with two components. Thus, all types of systems of parallel communicating finite automata accept more than regular languages. The rough idea of the construction is that in every time step the master component queries the non-master component, and the non-master component reads an input symbol. When the non-master component has read the separating symbol \$, which is notified to the master with the help of primed states, then the master component starts to compare its input symbol with the information from the non-master component. If all symbols up to \$ match, the input is accepted and in all other cases rejected. The precise construction is given through the following transition functions.

$$\begin{array}{lll}
\delta_1(s_{0,1}, \lambda) & = & q_2 \\
\delta_1(s_\$, \lambda) & = & q_2 \\
\delta_1(s_{\triangleleft}, \$) & = & \text{accept} \\
\delta_2(s_{0,2}, a) & = & s_a \\
\delta_2(s_a, a) & = & s_a \\
\delta_2(s_b, a) & = & s_a \\
\delta_2(s_\$, a) & = & s'_a \\
\delta_2(s'_a, a) & = & s'_a \\
\delta_2(s'_b, a) & = & s'_a \\
\delta_1(s_a, \lambda) & = & q_2 \\
\delta_1(s'_a, a) & = & q_2 \\
\delta_2(s_{0,2}, b) & = & s_b \\
\delta_2(s_a, b) & = & s_b \\
\delta_2(s_b, b) & = & s_b \\
\delta_2(s_\$, b) & = & s'_b \\
\delta_2(s'_a, b) & = & s'_b \\
\delta_2(s'_b, b) & = & s'_b \\
\delta_1(s_b, \lambda) & = & q_2 \\
\delta_1(s'_b, b) & = & q_2 \\
\delta_2(s_a, \$) & = & s_\$ \\
\delta_2(s_b, \$) & = & s_\$ \\
\delta_2(s_{\triangleleft}, \triangleleft) & = & s_{\triangleleft} \\
\delta_2(s'_a, \triangleleft) & = & s_{\triangleleft} \\
\delta_2(s'_b, \triangleleft) & = & s_{\triangleleft}
\end{array}$$

For nondeterministic non-centralized devices it is shown in [2] that returning parallel communicating finite automata systems are neither weaker nor stronger than non-returning ones. The question whether the same equivalence is true in the deterministic case was answered in the affirmative in [1]. To this end, the so-called *cycling-token-method* is introduced. Basically, the main problem of that method was to break the synchronization at the beginning. Otherwise, when some component  $A_{i+1}$  requests the state of  $A_i$  and, thus,  $A_i$  reenters its initial state, then  $A_i$  will request the state of  $A_{i-1}$  and so on. But these cascading communication requests would destroy necessary information. The next lemma is shown by applying the cycling-token-method.

**Lemma 33.** Let  $k \geq 1$ . Then  $\mathcal{L}(\text{DRPCFA}(k))$  includes  $\mathcal{L}(\text{DPCFA}(k))$ .

One of the fundamental results obtained in [25] is the characterization of the computational power of (unrestricted) parallel communicating finite automata systems by multi-head finite automata.

**Theorem 34.** Let  $k \geq 1$ . Then the families  $\mathcal{L}(\text{DRPCFA}(k))$ ,  $\mathcal{L}(\text{DPCFA}(k))$ , and  $\mathcal{L}(\text{1DFA}(k))$  are equal.

*Proof.* It remains to be shown that, for all  $k \geq 1$ , the family  $\mathcal{L}(\text{1DFA}(k))$  includes  $\mathcal{L}(\text{DRPCFA}(k))$ . Given some  $\text{DRPCFA}(k)$   $\mathcal{A}$ , basically, the idea of simulating it by a deterministic one-way  $k$ -head finite automaton  $A'$  is to track all current states of the components of  $\mathcal{A}$  in the current state of  $A'$  (see [25]).  $\square$

Comparing deterministic centralized systems with non-centralized systems we obtain for centralized systems the surprising result that the returning mode is not weaker than the non-returning mode. Let  $L_{\text{rc}} = \{uc^xv\$uv \mid u, v \in \{a,b\}^*, x \geq 0\}$ .

**Theorem 35.** The language  $L_{\text{rc}}$  belongs to the family  $\mathcal{L}(\text{DRCPCFA})$  (and thus to  $\mathcal{L}(\text{DRPCFA}) = \mathcal{L}(\text{DPCFA})$ ), but not to  $\mathcal{L}(\text{DCPCFA})$ .

**Corollary 36.**  $\mathcal{L}(\text{DCPCFA}) \subset \mathcal{L}(\text{DPCFA}) = \mathcal{L}(\text{DRPCFA})$ .

In order to show that nondeterministic centralized systems are strictly more powerful than their deterministic variants we consider the language  $L_{\text{mi}} = \{ww^R \mid w \in \{a,b,c\}^+\}$  and show that its complement belongs to  $\mathcal{L}(\text{CPCFA})$ , but does not belong to  $\mathcal{L}(\text{DPCFA})$ .

**Corollary 37.**  $\mathcal{L}(\text{DCPCFA}) \subset \mathcal{L}(\text{CPCFA})$  and  $\mathcal{L}(\text{DPCFA}) \subset \mathcal{L}(\text{PCFA})$ .

Finally, we compare the classes under consideration with some well-known language families.

**Lemma 38.** *The family  $\mathcal{L}(\text{PCFA})$  is strictly included in NL, hence, in the family of deterministic context-sensitive languages.*

*Proof.* Since  $L = \{ww^R \mid w \in \{a,b,c\}^+\}$  can be accepted by some 2NFA(2), language  $L$  belongs to NL. On the other hand, language  $L$  does not belong to  $\mathcal{L}(\text{1NFA}(k))$ , for all  $k \geq 1$ .  $\square$

**Lemma 39.** *All language classes accepted by parallel communicating finite automata systems are incomparable to the class of (deterministic) (linear) context-free languages.*

*Proof.* The language  $\{w\$w^R \mid w \in \{a,b\}^+\}$  of marked palindromes of even lengths is deterministic linear context free, but is not accepted by any 1NFA( $k$ ), for all  $k \geq 1$ . Thus, the language  $\{w\$w^R \mid w \in \{a,b\}^+\}$  does not belong to  $\mathcal{L}(\text{PCFA})$ . Conversely, the set  $\{w\$w \mid w \in \{a,b\}^+\}$  belongs to  $\mathcal{L}(\text{DRCPCFA})$  as well as to  $\mathcal{L}(\text{DCPCFA})$ , but is not context free.  $\square$

**Lemma 40.** *All language classes accepted by parallel communicating finite automata systems are incomparable with the class of Church-Rosser languages.*

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